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Probabilistic Theory of Mean Field Games with Applications

Mean Field Games with Common Noise and Master Equations

Springer
Volume I

Part I The Probabilistic Approach to Mean Field Games

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