

I. S. BEREZIN and N. P. ZHIDKOV

COMPUTING METHODS

VOLUME I

Translated by
O. M. BLUNN

Translation edited by

A. D. BOOTH
College of Engineering
University of Saskatchewan

PERGAMON PRESS

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