MARCEL BERGER

Geometry Revealed

A Jacob's Ladder to Modern Higher Geometry



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The geodesics, the cut locus and the recalcitrant ellipsoids

An indispensable abstract concept: Riemannian surfaces

Problems of isometries: abstract surfaces versus surfaces of \mathbb{E}^3 ...

Local shape of surfaces: the second fundamental form, total curvature and mean curvature, their geometric interpretation, the *theorema egregium*, the manufacture of precise balls

What is known about the total curvature (of Gauss)

What we don't entirely know how to do for surfaces

Convex functions of several variables, an important example

of the isoperimetric inequality and other applications

Volume and area of (compacts) convex sets, classical volumes: Can the volume be calculated in polynomial time?

Volume, area, diameter and symmetrizations: first proof

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