

*Sources in the Development  
of Mathematics*

*Infinite Series and Products from the  
Fifteenth to the Twenty-first Century*

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