

Eleuterio F. Toro

# Riemann Solvers and Numerical Methods for Fluid Dynamics

A Practical Introduction

Third Edition

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