

O. A. Bauchau

# Flexible Multibody Dynamics

 Springer

---

# Contents

---

## Part I Basic tools and concepts

---

<b>1</b>	<b>Vectors and tensors</b> .....	3
1.1	Free vectors .....	3
1.1.1	Vector sum .....	4
1.1.2	Scalar multiplication .....	4
1.1.3	Norm of a free vector .....	4
1.1.4	Angle between two vectors .....	5
1.1.5	The scalar product .....	5
1.1.6	Orthonormal bases .....	6
1.1.7	The vector product .....	7
1.1.8	The tensor product .....	9
1.1.9	The mixed product .....	10
1.1.10	Tensor identities .....	10
1.1.11	Solution of the vector product equation .....	11
1.1.12	Problems .....	12
1.2	Bound vectors .....	13
1.2.1	The position vector .....	14
1.2.2	Reference frames .....	14
1.3	Geometric entities .....	15
1.3.1	Lines .....	15
1.3.2	Planes .....	16
1.3.3	Circles .....	16
1.3.4	Spheres .....	16
1.3.5	Problems .....	20
1.4	Second-order tensors .....	22
1.4.1	Basic operations .....	22
1.4.2	Eigenvalue analysis .....	23
1.4.3	Problems .....	27
1.5	Tensor calculus .....	27

1.6	Notational conventions . . . . .	29
<b>2</b>	<b>Coordinate systems . . . . .</b>	<b>31</b>
2.1	Cartesian coordinates . . . . .	31
2.2	Differential geometry of a curve . . . . .	32
2.2.1	Intrinsic parameterization . . . . .	32
2.2.2	Arbitrary parameterization . . . . .	34
2.3	Path coordinates . . . . .	39
2.3.1	Problems . . . . .	39
2.4	Differential geometry of a surface . . . . .	40
2.4.1	The first metric tensor of a surface . . . . .	40
2.4.2	Curve on a surface . . . . .	41
2.4.3	The second metric tensor of a surface . . . . .	42
2.4.4	Analysis of curvatures . . . . .	43
2.4.5	Lines of curvature . . . . .	44
2.4.6	Derivatives of the base vectors . . . . .	44
2.4.7	Problems . . . . .	48
2.5	Surface coordinates . . . . .	49
2.6	Differential geometry of a three-dimensional mapping . . . . .	49
2.6.1	Arbitrary parameterization . . . . .	50
2.6.2	Orthogonal parameterization . . . . .	51
2.6.3	Derivatives of the base vectors . . . . .	52
2.7	Orthogonal curvilinear coordinates . . . . .	53
2.7.1	Cylindrical coordinates . . . . .	54
2.7.2	Spherical coordinates . . . . .	55
<b>3</b>	<b>Basic principles . . . . .</b>	<b>57</b>
3.1	Newtonian mechanics for a particle . . . . .	57
3.1.1	Kinematics of a particle . . . . .	57
3.1.2	Newton's laws . . . . .	58
3.1.3	Systems of units . . . . .	60
3.1.4	The principle of work and energy . . . . .	61
3.2	Conservative forces . . . . .	62
3.2.1	Principle of conservation of energy . . . . .	65
3.2.2	Potential of common conservative forces . . . . .	66
3.2.3	Non-conservative forces . . . . .	70
3.2.4	The principle of impulse and momentum . . . . .	74
3.2.5	Problems . . . . .	79
3.3	Contact forces . . . . .	85
3.3.1	Kinematics of particles in contact with a surface . . . . .	86
3.3.2	Kinematics of particles in contact with a curve . . . . .	87
3.3.3	Constitutive laws for tangential contact forces . . . . .	88
3.3.4	Problems . . . . .	92
3.4	Newtonian mechanics for a system of particles . . . . .	94
3.4.1	The center of mass . . . . .	95

3.4.2	The forces and moments . . . . .	96
3.4.3	Linear and angular momenta . . . . .	97
3.4.4	Euler's laws for a system of particles . . . . .	98
3.4.5	The principle of work and energy . . . . .	100
3.4.6	The principle of impulse and momentum . . . . .	100
3.4.7	Problems . . . . .	101
<b>4</b>	<b>The geometric description of rotation . . . . .</b>	<b>107</b>
4.1	The direction cosine matrix . . . . .	107
4.2	Planar rotations . . . . .	108
4.3	Non-commutativity of rotations . . . . .	109
4.4	Euler angles . . . . .	109
4.4.1	Problems . . . . .	111
4.5	Euler's theorem on rotations . . . . .	112
4.6	The rotation tensor . . . . .	113
4.7	Properties of the rotation tensor . . . . .	115
4.8	Change of basis operations . . . . .	116
4.8.1	Vector components in various orthonormal bases . . . . .	116
4.8.2	Second-order tensor components in various orthonormal bases . . . . .	117
4.8.3	Tensor operations . . . . .	119
4.8.4	The concept of tensor analysis . . . . .	121
4.8.5	Problems . . . . .	122
4.9	Composition of rotations . . . . .	124
4.9.1	Problems . . . . .	127
4.10	Time derivatives of rotation operations . . . . .	129
4.10.1	The angular velocity vector: an intuitive approach . . . . .	129
4.10.2	The angular velocity vector: a rigorous approach . . . . .	131
4.10.3	The addition theorem . . . . .	135
4.10.4	Angular acceleration . . . . .	137
4.11	Euler angle formulas . . . . .	137
4.11.1	Euler angles: sequence 3-1-3 . . . . .	138
4.11.2	Euler angles: sequence 3-2-3 . . . . .	139
4.11.3	Euler angles: sequence 3-2-1 . . . . .	139
4.11.4	Euler angles: sequence 3-1-2 . . . . .	140
4.11.5	Problems . . . . .	141
4.12	Spatial derivatives of rotation operations . . . . .	143
4.12.1	Path coordinates . . . . .	143
4.12.2	Surface coordinates . . . . .	144
4.12.3	Orthogonal curvilinear coordinates . . . . .	145
4.12.4	The differential rotation vector . . . . .	148
4.13	Applications to particle dynamics . . . . .	149
4.13.1	Problems . . . . .	152
4.14	Change of reference frame operations . . . . .	155
4.15	Orientation of a unit vector . . . . .	157

---

**Part II Rigid body dynamics**


---

<b>5</b>	<b>Kinematics of rigid bodies</b> . . . . .	161
5.1	General motion of a rigid body . . . . .	161
5.2	Velocity field of a rigid body . . . . .	163
5.2.1	Problems . . . . .	165
5.3	Relative velocity and acceleration . . . . .	166
5.3.1	Point $\mathbf{P}$ is in motion with respect to the rigid body . . . . .	167
5.3.2	Point $\mathbf{P}$ is a material point of the rigid body . . . . .	168
5.3.3	Problems . . . . .	174
5.4	Contact between rigid bodies . . . . .	179
5.4.1	Problems . . . . .	183
5.5	The motion tensor . . . . .	187
5.5.1	Transformation of a line of a rigid body . . . . .	187
5.5.2	Properties of the motion tensor . . . . .	188
5.5.3	Mozzi-Chasles' axis . . . . .	189
5.5.4	Intrinsic expression of the motion tensor . . . . .	190
5.5.5	Properties of the generalized vector product tensor . . . . .	191
5.5.6	Change of frame operation for linear and angular velocities . . . . .	192
5.5.7	Change of frame operation for forces and moments . . . . .	193
5.6	Derivatives of finite motion operations . . . . .	194
5.6.1	The velocity vector . . . . .	194
5.6.2	The differential motion vector . . . . .	196
5.6.3	Change of frame operations . . . . .	197
<b>6</b>	<b>Kinetics of rigid bodies</b> . . . . .	201
6.1	The angular momentum vector . . . . .	202
6.2	The kinetic energy . . . . .	204
6.3	Properties of the mass moment of inertia tensor . . . . .	205
6.3.1	The parallel axis theorem . . . . .	205
6.3.2	Change of basis . . . . .	206
6.3.3	Principal axes of inertia . . . . .	207
6.3.4	Problems . . . . .	208
6.4	Derivatives of the angular momentum vector . . . . .	210
6.5	Equations of motion for a rigid body . . . . .	211
6.5.1	Euler's equations . . . . .	212
6.5.2	The pivot equations . . . . .	213
6.5.3	Equations of motion with respect to a material point of the rigid body . . . . .	213
6.5.4	Equations of motion with respect to an arbitrary point . . . . .	214
6.6	The principle of work and energy . . . . .	214
6.6.1	Problems . . . . .	222
6.7	Planar motion of rigid bodies . . . . .	227
6.7.1	Euler's equations . . . . .	227

6.7.2	The pivot equations . . . . .	228
6.7.3	Equations of motion with respect to a material point of the body . . . . .	228
6.7.4	Equations of motion with respect to an arbitrary point . . . . .	229
6.7.5	Problems . . . . .	237
6.8	Inertial characteristics . . . . .	248

**Part III Concepts of analytical dynamics**

<b>7</b>	<b>Basic concepts of analytical dynamics . . . . .</b>	<b>253</b>
7.1	Mathematical preliminaries . . . . .	253
7.1.1	Stationary point of a function . . . . .	254
7.1.2	Stationary point of a definite integral . . . . .	255
7.2	Generalized coordinates . . . . .	257
7.3	The virtual displacement and rotation vectors . . . . .	263
7.3.1	Problems . . . . .	268
7.4	Virtual work and generalized forces . . . . .	268
7.4.1	Virtual work . . . . .	268
7.4.2	Generalized forces . . . . .	268
7.4.3	Virtual work done by internal forces . . . . .	269
7.4.4	Problems . . . . .	271
7.5	The principle of virtual work for statics . . . . .	271
7.5.1	Principle of virtual work for a single particle . . . . .	272
7.5.2	Kinematically admissible virtual displacements . . . . .	277
7.5.3	Use of infinitesimal displacements as virtual displacements . . . . .	282
7.5.4	Principle of virtual work for a system of particles . . . . .	284
7.5.5	The use of generalized coordinates . . . . .	287
7.5.6	The principle of virtual work and conservative forces . . . . .	289
7.5.7	Problems . . . . .	292
<b>8</b>	<b>Variational and energy principles . . . . .</b>	<b>295</b>
8.1	D'Alembert's principle . . . . .	295
8.1.1	Equations of motion for a rigid body . . . . .	297
8.1.2	Equations of motion for the planar motion of a rigid body . . . . .	299
8.1.3	Problems . . . . .	304
8.2	Hamilton's principle . . . . .	305
8.2.1	Use of physical coordinates . . . . .	306
8.2.2	Use of generalized coordinates . . . . .	308
8.2.3	Problems . . . . .	320
8.3	Lagrange's formulation . . . . .	322
8.3.1	Problems . . . . .	335
8.4	Analysis of the motion . . . . .	336
8.4.1	General procedure for the analysis of motion . . . . .	337
8.4.2	Problems . . . . .	345

---

**Part IV Constrained dynamical systems**


---

<b>9</b>	<b>Constrained systems: preliminaries</b> .....	351
9.1	Lagrange's multiplier method .....	358
9.1.1	Problems .....	362
9.2	Constraints .....	362
9.2.1	Holonomic constraints .....	362
9.2.2	Nonholonomic constraints .....	364
9.2.3	Problems .....	370
9.3	The principle of virtual work for constrained static problems .....	371
9.3.1	The principle of virtual work for a constrained particle .....	372
9.3.2	The principle of virtual work and Lagrange multipliers .....	378
9.3.3	Problems .....	382
<b>10</b>	<b>Constrained systems: classical formulations</b> .....	385
10.1	D'Alembert's principle for constrained systems .....	385
10.1.1	Problems .....	390
10.2	Hamilton's principle and Lagrange's formulation with holonomic constraints .....	392
10.2.1	Hamilton's principle with holonomic constraints .....	393
10.2.2	Lagrange's formulation with holonomic constraints .....	394
10.2.3	Problems .....	398
10.3	Hamilton's principle and Lagrange's formulation with nonholonomic constraints .....	402
10.3.1	Hamilton's principle with nonholonomic constraints .....	403
10.3.2	Lagrange's formulation with nonholonomic constraints .....	403
10.3.3	Problems .....	405
10.4	The lower pair joints .....	405
10.4.1	Kinematics of a typical lower pair joint .....	406
10.4.2	Notational conventions .....	406
10.4.3	Relative motions .....	407
10.5	Generic constraints for lower pair joints .....	408
10.5.1	First constraint: vanishing relative rotation .....	409
10.5.2	Second constraint: vanishing relative displacement .....	410
10.5.3	Third constraint: definition of relative rotation .....	411
10.5.4	Fourth constraint: definition of relative displacement .....	411
10.6	Constraints for the lower pair joints .....	412
10.6.1	Revolute joints .....	412
10.6.2	Prismatic joints .....	414
10.6.3	Cylindrical joints .....	415
10.6.4	Screw joints .....	416
10.6.5	Planar joints .....	416
10.6.6	Spherical joints .....	417
10.6.7	Problems .....	418

10.7	Other joints	418
10.7.1	Universal joints	418
10.7.2	Curve sliding joints	419
10.7.3	Sliding joints	421
10.7.4	Problems	422
<b>11</b>	<b>Constrained systems: advanced formulations</b>	<b>425</b>
11.1	Lagrange's equations of the first kind	426
11.2	Algebraic elimination of Lagrange's multipliers	427
11.2.1	Maggi's formulation	428
11.2.2	Problems	435
11.2.3	Index-1 formulation	438
11.2.4	Problems	441
11.2.5	The null space formulation	442
11.2.6	Problems	444
11.2.7	Udwadia and Kalaba's formulation	444
11.2.8	Comparison of the ODE formulations	445
11.2.9	Problems	447
11.3	The geometric interpretation of constraints	447
11.3.1	The orthogonal projection operator	448
11.3.2	The projection operator	450
11.3.3	Projection of the equations of motion	453
11.3.4	Elimination of Lagrange's multipliers	455
11.4	Gauss' principle	457
11.5	Additional formulations	461
<b>12</b>	<b>Constrained systems: numerical methods</b>	<b>463</b>
12.1	Ordinary differential equation techniques	464
12.1.1	"Maggi-like" formulations	464
12.1.2	Maggi's formulations	466
12.1.3	Discussion of the methods based on Maggi's formulation	467
12.1.4	Null space formulations	468
12.1.5	Udwadia and Kalaba's formulations	469
12.1.6	The projective formulation	469
12.1.7	Modified phase space formulation	470
12.2	Index reduction techniques	470
12.3	Constraint violation stabilization techniques	473
12.3.1	Control theory based stabilization techniques	473
12.3.2	Penalty based stabilization techniques	475
12.4	Constraint violation elimination techniques	478
12.4.1	Geometric projection approach to stabilization	478
12.4.2	The mass-orthogonal projection formulation	480
12.5	Finite element based techniques	481
12.5.1	Floating frame of reference approach	482
12.5.2	Component mode synthesis methods	484



12.5.3	Basic solution techniques for finite element models . . . . .	485
12.5.4	Numerically dissipative schemes . . . . .	487
12.5.5	Nonlinear unconditionally stable schemes . . . . .	487
12.5.6	Enforcement of the constraints . . . . .	488
12.5.7	The discrete null space approach . . . . .	489
12.6	Scaling of Lagrange's equation of the first kind . . . . .	490
12.6.1	Scaling of the equations of motion . . . . .	492
12.6.2	The augmented Lagrangian term . . . . .	493
12.6.3	Time discretization of the equations . . . . .	494
12.6.4	Relationship to the preconditioning approach . . . . .	498
12.6.5	Benefits of the augmented Lagrangian formulation . . . . .	499
12.6.6	Using other time integration schemes . . . . .	501
12.7	Conclusions . . . . .	504

---

## Part V Parameterization of rotation and motion

---

<b>13</b>	<b>Parameterization of rotation . . . . .</b>	<b>511</b>
13.1	Cayley's rotation parameters . . . . .	513
13.2	Quaternion algebra . . . . .	514
13.3	Euler parameters . . . . .	516
13.3.1	The rotation tensor . . . . .	517
13.3.2	The angular velocity vector . . . . .	517
13.3.3	Composition of rotations . . . . .	518
13.3.4	Determination of Euler parameters . . . . .	518
13.3.5	Problems . . . . .	523
13.4	The vectorial parameterization of rotation . . . . .	524
13.4.1	Fundamental properties . . . . .	524
13.4.2	The rotation tensor . . . . .	526
13.4.3	The angular velocity vector . . . . .	527
13.4.4	Determination of the rotation parameter vector . . . . .	529
13.4.5	Composition of rotations . . . . .	530
13.4.6	Linearization of the tangent tensor . . . . .	530
13.4.7	Problems . . . . .	530
13.5	Specific choices of generating function . . . . .	531
13.6	The extended vectorial parameterization . . . . .	533
13.6.1	Singularities of the vectorial parameterization . . . . .	533
13.6.2	The rescaling operation . . . . .	534
13.7	Specific parameterizations of rotation . . . . .	537
13.7.1	The Cartesian rotation vector . . . . .	537
13.7.2	The Euler-Rodrigues parameters . . . . .	538
13.7.3	The Cayley-Gibbs-Rodrigues parameters . . . . .	538
13.7.4	The Wiener-Milenković parameters . . . . .	539
13.7.5	Problems . . . . .	541

<b>14</b>	<b>Parameterization of motion</b> .....	543
14.1	Cayley's motion parameters .....	544
14.2	Bi-quaternion algebra .....	546
14.3	Euler motion parameters .....	547
14.3.1	The motion tensor .....	548
14.3.2	The velocity vector .....	549
14.3.3	Composition of finite motions .....	550
14.3.4	Determination of Euler motion parameters .....	550
14.3.5	Problems .....	553
14.4	The vectorial parameterization of motion .....	554
14.4.1	Fundamental properties .....	554
14.4.2	The motion parameter vector .....	556
14.4.3	The generalized vector product tensor .....	558
14.4.4	The motion tensor .....	558
14.4.5	The velocity vector .....	559
14.4.6	Determination of the motion parameter vector .....	561
14.4.7	Composition of finite motions .....	561
14.5	Specific parameterizations of motion .....	561
14.5.1	Alternative choices of the motion parameter vector .....	562
14.5.2	The exponential map of motion .....	563
14.5.3	The Euler-Rodrigues motion parameters .....	563
14.5.4	The Cayley-Gibbs-Rodrigues motion parameters .....	563
14.5.5	The Wiener-Milenković motion parameters .....	564
14.5.6	Problems .....	564

---

**Part VI Flexible multibody dynamics**

---

<b>15</b>	<b>Flexible multibody systems: preliminaries</b> .....	569
15.1	Classification of multibody systems .....	569
15.1.1	Linearly and nonlinearly elastic multibody systems .....	570
15.1.2	Shortcomings of modal analysis applied to nonlinear systems .....	571
15.1.3	Finite element based modeling of flexible multibody systems .....	575
15.2	The elastodynamics problem .....	579
15.2.1	Review of the equations of linear elastodynamics .....	580
15.2.2	The principle of virtual work .....	583
15.2.3	Hamilton's principle .....	586
15.3	Finite displacements kinematics for flexible bodies .....	588
15.3.1	The engineering strain components .....	592
15.3.2	The deformation gradient tensor .....	592
15.3.3	The metric tensor .....	593
15.3.4	The Green-Lagrange strain tensor .....	593
15.4	Strain measures for various differential elements .....	594
15.4.1	Stretch of a material line .....	594
15.4.2	Angle between two material lines .....	595

- 15.4.3 Surface dilatation . . . . . 595
- 15.4.4 Volume dilatation . . . . . 596
- 15.4.5 Problems . . . . . 596
- 15.5 The formulation of small strain problems . . . . . 597
  - 15.5.1 Decomposition of the deformation gradient tensor . . . . . 597
  - 15.5.2 The small strain assumption . . . . . 598
- 16 Formulation of flexible elements . . . . . 601**
  - 16.1 Formulation of flexible joints . . . . . 601
    - 16.1.1 Flexible joint configuration . . . . . 603
    - 16.1.2 Flexible joint differential work . . . . . 604
    - 16.1.3 The deformation measures . . . . . 605
    - 16.1.4 Change of reference frame . . . . . 606
    - 16.1.5 Deformation measure invariance . . . . . 607
    - 16.1.6 Flexible joint constitutive laws . . . . . 608
  - 16.2 Formulation of cable equations . . . . . 613
    - 16.2.1 The kinematics of the problem . . . . . 613
    - 16.2.2 The small strain assumption . . . . . 614
    - 16.2.3 Governing equations . . . . . 615
    - 16.2.4 Extension to dynamic problems . . . . . 616
    - 16.2.5 Problems . . . . . 616
  - 16.3 Formulation of beam equations . . . . . 617
    - 16.3.1 Kinematics of the problem . . . . . 618
    - 16.3.2 Governing equations . . . . . 620
    - 16.3.3 Extension to dynamic problems . . . . . 622
    - 16.3.4 Problems . . . . . 628
  - 16.4 Formulation of plate and shell equations . . . . . 628
    - 16.4.1 Kinematics of the shell problem . . . . . 629
    - 16.4.2 Governing equations . . . . . 631
    - 16.4.3 Extension to dynamic problems . . . . . 633
    - 16.4.4 Mixed interpolation of tensorial components . . . . . 634
- 17 Finite element tools . . . . . 639**
  - 17.1 Interpolation of displacement fields . . . . . 639
  - 17.2 Interpolation of rotation fields . . . . . 644
    - 17.2.1 Finite element discretization . . . . . 646
    - 17.2.2 Total versus incremental unknowns . . . . . 650
    - 17.2.3 Interpolation of incremental rotations . . . . . 651
  - 17.3 Governing equations and linearization process . . . . . 657
    - 17.3.1 Statics problems . . . . . 657
    - 17.3.2 Problems . . . . . 659
    - 17.3.3 Linear structural dynamics problems . . . . . 660
    - 17.3.4 Nonlinear structural dynamics problems . . . . . 661
    - 17.3.5 Multibody dynamics problems with holonomic constraints . . . . . 662
    - 17.3.6 Multibody dynamics problems with nonholonomic constraints . . . . . 663

17.4	The generalized- $\alpha$ time integration scheme . . . . .	664
17.4.1	Linear structural dynamics problems . . . . .	665
17.4.2	Nonlinear structural dynamics problems . . . . .	667
17.4.3	Multibody dynamics problems with holonomic constraints . . . . .	669
17.4.4	Multibody dynamics problems with nonholonomic constraints . . . . .	669
17.5	Energy preserving and decaying schemes . . . . .	670
17.5.1	The symmetric hyperbolic form . . . . .	672
17.5.2	Discussion . . . . .	677
17.5.3	Practical time integration schemes . . . . .	677
17.5.4	Enforcement of the constraints . . . . .	682
17.6	Implementation of cable elements . . . . .	685
17.6.1	Inertial forces . . . . .	686
17.6.2	Elastic forces . . . . .	686
17.6.3	Dissipative forces . . . . .	687
17.6.4	Gravity forces for cables . . . . .	687
17.6.5	Finite element formulation of cables . . . . .	688
17.7	Finite element implementation of beam elements . . . . .	689
17.7.1	Inertial forces . . . . .	689
17.7.2	Elastic forces . . . . .	691
17.7.3	Dissipative forces . . . . .	693
17.7.4	Gravity forces for beams . . . . .	695
17.7.5	Finite element formulation of beams . . . . .	695
<b>18</b>	<b>Mathematical tools . . . . .</b>	<b>697</b>
18.1	The singular value decomposition . . . . .	697
18.2	The Moore-Penrose generalized inverse . . . . .	698
18.2.1	Problems . . . . .	699
18.3	Gauss-Legendre quadrature . . . . .	699
<b>References . . . . .</b>		<b>703</b>
<b>Index . . . . .</b>		<b>721</b>