# The Early Mathematics of Leonhard Euler

C. Edward Sandifer
*Western Connecticut State University*

*Published and Distributed by*
THE MATHEMATICAL ASSOCIATION OF AMERICA

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The asterisks in front of the chapter titles below represent the author's rating system: the more asterisks, the more important the article.


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